

2-LOCAL TRIPLE HOMOMORPHISMS ON VON NEUMANN ALGEBRAS AND JBW*-TRIPLES

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ABSTRACT. We prove that every (not necessarily linear nor continuous) 2-local triple homomorphism from a JBW*-triple into a JB*-triple is linear and a triple homomorphism. Consequently, every 2-local triple homomorphism from a von Neumann algebra (respectively, from a JBW*-algebra) into a C*-algebra (respectively, into a JB*-algebra) is linear and a triple homomorphism.

1. INTRODUCTION

It is known that the Gleason-Kahane-Żelazko theorem (cf. [23, 28, 45]) admits a reinterpretation affirming that every unital linear local homomorphism from a unital complex Banach algebra A into \mathbb{C} is multiplicative. Formally speaking, the notions of local homomorphisms and local derivations were introduced in 1990, in papers due to Larson and Sourour [34] and Kadison [27]. We recall that given two Banach algebras A and B , a linear mapping $T : A \rightarrow B$ (respectively, $T : A \rightarrow A$) is said to be a *local homomorphism* (respectively, a *local derivation*) if for every a in A there exists a homomorphism $\Phi_a : A \rightarrow B$ (respectively, a derivation $D_a : A \rightarrow A$), depending on a , satisfying $T(a) = \Phi_a(a)$ (respectively, $T(a) = D_a(a)$). A flourishing research on linear local homomorphisms and derivations was built upon the results of Kadison, Larson and Sourour (compare, for example, [1, 5, 7, 8, 13, 15, 17, 18, 20, 26, 31, 32], [36]–[44] and [46], among the over 100 references on the subject).

If in the definition of *local homomorphism*, we relax the assumption concerning linearity with a 2-local behavior, we are led to the notion of (not necessarily linear) *2-local homomorphism*. Let A and B be two C*-algebras, a not necessarily linear nor continuous mapping $T : A \rightarrow B$ is said to be a *2-local homomorphism* (respectively, *2-local *-homomorphism*) if for

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every $a, b \in A$ there exists a bounded (linear) homomorphism (respectively, $*$ -homomorphism) $\Phi_{a,b} : A \rightarrow B$, depending on a and b , such that $\Phi_{a,b}(a) = T(a)$ and $\Phi_{a,b}(b) = T(b)$ (see [43], [14]).

In a recent contribution, we establish a generalization of the Kowalski-Słodkowski theorem for 2-local $*$ -homomorphisms on von Neumann algebras, showing that every (not necessarily linear nor continuous) 2-local $*$ -homomorphism from a von Neumann algebra or from a compact C^* -algebra into a C^* -algebra is linear and a $*$ -homomorphism. In the Jordan setting, it is proved that every 2-local Jordan $*$ -homomorphism from a JBW^* -algebra into a JB^* -algebra is linear and a Jordan $*$ -homomorphism (cf. [14]).

Every C^* -algebra A admits a ternary product given by

$$\{a, b, c\} := \frac{1}{2}(ab^*c + cb^*a) \quad (a, b, c \in A).$$

A linear map Φ between C^* -algebras A and B satisfying $\Phi(\{a, b, c\}) = \{\Phi(a), \Phi(b), \Phi(c)\}$, is called a *triple homomorphism*. A *2-local triple homomorphism* between A and B is a not necessarily linear nor continuous map $T : A \rightarrow B$ such that for every $a, b \in A$, there exists a triple homomorphism $\Phi_{a,b} : A \rightarrow B$ with $\Phi_{a,b}(a) = T(a)$ and $\Phi_{a,b}(b) = T(b)$. Motivated by the above commented Kowalski-Słodkowski theorem for von Neumann algebras, it seems natural to consider the following independent problem:

Problem 1.1. *Is every 2-local triple homomorphism between C^* -algebras (automatically) linear?*

It should be noted here that, even in the case of von Neumann algebras, the proofs and arguments given in the study of 2-local $*$ -homomorphisms [14], are no longer valid when considering Problem 1.1, because triple homomorphisms between C^* -algebras do not preserve the natural partial order given by the positive cone in a C^* -algebra.

Problem 1.1 can be posed in the more general setting of JB^* -triples. Let E and F be two JB^* -triples (see subsection 1.1 for definitions). A linear map $\Phi : E \rightarrow F$ which preserves the triple products is called a *triple homomorphism*. A (not necessarily linear nor continuous) mapping $T : E \rightarrow F$ is said to be a *2-local triple homomorphism* if for every $a, b \in E$ there exists a bounded (linear) triple homomorphism $\Phi_{a,b} : E \rightarrow F$, depending on a and b , such that $\Phi_{a,b}(a) = T(a)$ and $\Phi_{a,b}(b) = T(b)$. According to these definitions, we consider the following generalization of Problem 1.1:

Problem 1.2. *Is every 2-local triple homomorphism between JB^* -triples (automatically) linear?*

In this paper we solve Problems 1.1 and 1.2 when the domain is a von Neumann algebra or a JBW^* -triple, respectively. Our main result (Theorem 3.8) asserts that every (not necessarily linear nor continuous) 2-local triple homomorphism from a JBW^* -triple into a JB^* -triple is linear and

a triple homomorphism, and consequently, every 2-local triple homomorphism from a von Neumann algebra (respectively, from a JBW*-algebra) into a C*-algebra (respectively, into a JB*-algebra) is linear and a triple homomorphism (cf. Theorem 3.5 and Corollary 3.6). Our proofs heavily rely on the Bunce-Wright-Mackey-Gleason theorem for JBW*-algebras [9] and deep geometric arguments and techniques, developed in the setting of JB*-triples by R. Braun, W. Kaup and H. Upmeyer [6, 30], B. Russo and Y. Friedman [22], and G. Horn [25].

1.1. Preliminaries. A *JB*-triple* is a complex Banach space, E , together with a continuous triple product $\{.,.,.\} : E \times E \times E \rightarrow E$, $(a, b, c) \mapsto \{a, b, c\}$, which is conjugate-linear in b and symmetric and bilinear in (a, c) and satisfies:

(1) The *Jordan identity*:

$$L(a, b)L(x, y) - L(x, y)L(a, b) = L(L(a, b)x, y) - L(x, L(b, a)y),$$

where $L(a, b)$ denotes the operator given by $L(a, b)x = \{a, b, x\}$;

(2) $L(a, a)$ is an hermitian operator with non-negative spectrum;

(3) $\|\{a, a, a\}\| = \|a\|^3$,

every a, b, x and y in E .

The notion of JB*-triples was introduced by Kaup in the holomorphic classification of bounded symmetric domains in [29]. One of the many kindness exhibited by the class of JB*-triples is that every C*-algebra (respectively, every JB*-algebra) is a JB*-triple with respect to

$$\{a, b, c\} := \frac{1}{2}(ab^*c + cb^*a)$$

(respectively, $\{a, b, c\} := (a \circ b^*) \circ c + (c \circ b^*) \circ a - (a \circ c) \circ b^*$).

A *JBW*-triple* is a JB*-triple which is also a dual Banach space (with a unique isometric predual [4]). It is known that the triple product of a JBW*-triple is separately weak* continuous (cf. [4]).

We recall that an element e in a JB*-triple E is said to be a *tripotent* if $\{e, e, e\} = e$. It is known that for each tripotent e in E we have a decomposition (called the *Peirce decomposition*)

$$E = E_2(e) \oplus E_1(e) \oplus E_0(e),$$

where for $j = 0, 1, 2$, $E_j(e)$ is the $\frac{j}{2}$ -eigenspace of $L(e, e)$. The Peirce subspaces $E_j(e)$ satisfy the following multiplication rules:

$$\{E_i(e), E_j(e), E_k(e)\} \subseteq E_{i-j+k}(e),$$

if $i - j + k \in \{0, 1, 2\}$ and is zero otherwise, and

$$\{E_2(e), E_0(e), E\} = \{E_0(e), E_2(e), E\} = 0.$$

These multiplication rules are called the *Peirce rules*. The natural projection $P_j(e) : E \rightarrow E_j(e)$ of E onto $E_j(e)$ is called the *Peirce- j projection*. The

Peirce projections are contractive and satisfy

$$P_2(e) = L(e, e)(2L(e, e) - Id), \quad P_1(e) = 4L(e, e)(Id - L(e, e)),$$

$$\text{and } P_0(e) = (Id - L(e, e))(Id - 2L(e, e)),$$

where Id denotes the identity map on E (compare [22]). It is also known that for each $x_0 \in E_0(e)$ and $x_2 \in E_2(e)$ we have $\|x_0 + x_2\| = \max\{\|x_0\|, \|x_2\|\}$ (c.f. [22, Lemma 1.3]). The tripotent e is called *complete* when $E_0(e) = \{0\}$.

Another interesting property of the Peirce decomposition asserts that $E_2(e)$ is a unital JB*-algebra with unit e , product $a \circ_e b = \{a, e, b\}$ and involution $a^{\sharp_e} = \{e, a, e\}$ (c.f. [6, Theorem 2.2] and [30, Theorem 3.7]).

Accordingly to the standard terminology, for each element a in a JB*-triple E , we denote $a^{[1]} = a$ and $a^{[2n+1]} := \{a, a^{[2n-1]}, a\}$ ($\forall n \in \mathbb{N}$). It follows from the Jordan identity that JB*-triples are power associative, that is, $\{a^{[2k-1]}, a^{[2l-1]}, a^{[2m-1]}\} = a^{[2(k+l+m)-3]}$. In this paper, the symbol E_a will denote the JB*-subtriple of E generated by a . It is known that E_a is JB*-triple isomorphic (and hence isometric) to $C_0(L)$ for some locally compact Hausdorff space $L \subseteq (0, \|a\|]$, such that $L \cup \{0\}$ is compact and $\|a\| \in L$. It is further known that there exists a triple isomorphism Ψ from E_a onto $C_0(L)$, satisfying $\Psi(a)(t) = t$ ($t \in L$) (compare [29, Lemma 1.14]). In particular, for each natural n , there exists (a unique) element $a^{[1/(2n-1)]}$ in E_a satisfying $(a^{[1/(2n-1)]})^{[2n-1]} = a$.

When a is a norm one element in a JBW*-triple E , the sequence $(a^{[1/(2n-1)]})$ converges in the weak* topology of E to a tripotent in E , which is denoted by $r(a)$ and is called the *range tripotent* of a . The tripotent $r(a)$ is the smallest tripotent e in E satisfying that a is positive in the JBW*-algebra $E_2(e)$ (cf. [19, Lemma 3.3]).

We refer to [16] for a recent monograph on JB*-triples and JB*-algebras.

Throughout the paper, when A is a C*-algebra or a JB*-algebra, the symbol A_{sa} will stand for the set of all self-adjoint elements in A .

2. GENERALITIES ON 2-LOCAL TRIPLE HOMOMORPHISMS

We recall that elements a and b in a JB*-triple E are said to be *orthogonal* (written $a \perp b$) when $L(a, b) = 0$. It is known that $a \perp b$ if and only if $\{a, a, b\} = 0$, if and only if $\{a, b, b\} = 0$ (cf. [11, Lemma 1.1]). A pair of subsets $M, N \subset E$ are called orthogonal ($M \perp N$) if for every $a \in M, b \in N$, we have $a \perp b$.

Throughout the paper, given a 2-local triple homomorphism T between JB*-triples E and F , for each $a, b \in E$, $\Phi_{a,b}$ will denote a (linear) triple homomorphism satisfying $T(a) = \Phi_{a,b}(a)$ and $T(b) = \Phi_{a,b}(b)$.

We begin with some basic properties of 2-local triple homomorphisms.

Lemma 2.1. *Let $T : E \rightarrow F$ be a (not necessarily linear nor continuous) 2-local triple homomorphism between JB^* -triples. The following statements hold:*

- (a) *T is 1-homogeneous, that is, $T(\lambda a) = \lambda T(a)$ for every $a \in E$, $\lambda \in \mathbb{C}$;*
- (b) *T is orthogonality preserving;*
- (c) *$\{T(a), T(a), T(a)\} = T(\{a, a, a\})$, for every $a \in E$. In particular, every linear 2-local triple homomorphism between JB^* -triples is a triple homomorphism;*
- (d) *T maps tripotents in E to tripotents in F ;*
- (e) *For each $a, b \in E$, $\|T(a) - T(b)\| \leq \|a - b\|$, that is, T is 1-lipschitzian and hence continuous;*
- (f) *For each tripotent e in E with $T(e) \neq 0$, we have $T(E_j(e)) \subseteq F_j(T(e))$, for every $j = 0, 1, 2$, $T(E_2(e) + E_1(e)) \subseteq F_2(T(e)) + F_1(T(e))$, and $T(E_0(e) + E_1(e)) \subseteq F_0(T(e)) + F_1(T(e))$. Furthermore, $T(E_2(e)_{sa}) \subseteq F_2(T(e))_{sa}$;*
- (g) *For each tripotent $e \in E$ with $T(e) = 0$, the mapping T is zero on $E_2(e) \oplus E_1(e)$.*

Proof. The proof of (a) is standard (compare [14, Lemma 2.1]). For the statement (b), we recall that $a \perp b$ if and only if $\{a, a, b\} = 0$ [11, Lemma 1.1]. Let us consider the triple homomorphism $\Phi_{a,b} : E \rightarrow F$. Then

$$\{T(a), T(a), T(b)\} = \{\Phi_{a,b}(a), \Phi_{a,b}(a), \Phi_{a,b}(b)\} = \Phi_{a,b}\{a, a, b\} = 0,$$

which proves $T(a) \perp T(b)$.

(c) Considering the triple homomorphism $\Phi_{a,a^{[3]}}$, we have

$$\begin{aligned} \{T(a), T(a), T(a)\} &= \{\Phi_{a,a^{[3]}}(a), \Phi_{a,a^{[3]}}(a), \Phi_{a,a^{[3]}}(a)\} \\ &= \Phi_{a,a^{[3]}}(a^{[3]}) = T(\{a, a, a\}). \end{aligned}$$

The second statement follows from the polarization formula

$$8\{x, y, z\} = \sum_{k=0}^3 \sum_{j=1}^2 i^k (-1)^j \left(x + i^k y + (-1)^j z \right)^{[3]}.$$

(d) is clear from (c), and (e) follows from the fact that every triple homomorphism between JB^* -triples is contractive (cf. [3, Lemma 1] and the proof of [14, Lemma 2.1]).

(f) Let us take a tripotent $e \in E$ with $T(e)$ a non-zero tripotent in F . For each $a \in E_j(e)$ we have $L(e, e)(a) = \frac{j}{2}a$. Therefore,

$$\begin{aligned} L(T(e), T(e))T(a) &= \{T(e), T(e), T(a)\} = \{\Phi_{e,a}(e), \Phi_{e,a}(e), \Phi_{e,a}(a)\} \\ &= \Phi_{e,a}(\{e, e, a\}) = \frac{j}{2}\Phi_{e,a}(a) = \frac{j}{2}T(a), \end{aligned}$$

witnessing that $T(E_j(e)) \subseteq F_j(T(e))$, for every $j = 0, 1, 2$. We can similarly show that $T(a) \in \ker(Q(T(e))) = F_0(T(e)) + F_1(T(e))$ for every $a \in \ker(Q(e)) = E_0(e) + E_1(e)$ which shows that $T(E_0(e) + E_1(e)) \subseteq F_0(T(e)) + F_1(T(e))$.

Since $F_2(T(e)) + F_1(T(e)) = \ker(P_0(T(e)))$ and

$$P_0(T(e)) = (Id_F - L(T(e), T(e)))(Id_F - 2L(T(e), T(e))),$$

we can show, applying the triple homomorphism $\Phi_{a,e}$, that, for each element $a \in \ker(P_0(e)) = E_2(e) + E_1(e)$, we have $T(a) \in \ker(P_0(T(e)))$, which gives the other inclusion.

Suppose $a \in E_2(e)_{sa} = \{x \in E_2(e) : x = x^{\sharp e} = \{e, x, e\}\}$. Since

$$\begin{aligned} \{T(e), T(a), T(e)\} &= \{\Phi_{e,a}(e), \Phi_{e,a}(a), \Phi_{e,a}(e)\} \\ &= \Phi_{e,a}(\{e, a, e\}) = \Phi_{e,a}(a) = T(a), \end{aligned}$$

we deduce that $T(a) \in F_2(T(e))_{sa}$.

(g) Suppose $T(e) = 0$ and $a = a_1 + a_2$, where $a_j \in E_j(e)$ for $j = 1, 2$. In such a case

$$\begin{aligned} T(a) &= \Phi_{a,e}(a) = \Phi_{a,e}(\{e, e, a_2\}) + 2\Phi_{a,e}(\{e, e, a_1\}) \\ &= \{\Phi_{a,e}(e), \Phi_{a,e}(e), \Phi_{a,e}(a_2)\} + 2\{\Phi_{a,e}(e), \Phi_{a,e}(e), \Phi_{a,e}(a_1)\} \\ &= \{T(e), T(e), \Phi_{a,e}(a_2)\} + 2\{T(e), T(e), \Phi_{a,e}(a_1)\} = 0. \end{aligned}$$

□

We shall establish next a triple version of [14, Lemma 3.1].

Lemma 2.2. *Let $T : E \rightarrow F$ be a (not necessarily linear) 2-local triple homomorphism between JB^* -triples. Then, for each $a \in E$, $T|_{E_a} : E_a \rightarrow F$ is a linear mapping.*

Proof. Let us consider an element $b \in E_a$ of the form $b = \sum_{k=1}^m \alpha_k a^{[2k-1]}$ and the triple homomorphism $\Phi_{a,b}$. The identity

$$T(b) = \Phi_{a,b} \left(\sum_{k=1}^m \alpha_k a^{[2k-1]} \right) = \sum_{k=1}^m \alpha_k \Phi_{a,b}(a)^{[2k-1]} = \sum_{k=1}^m \alpha_k T(a)^{[2k-1]},$$

proves that T is linear on the linear span of the set $\{a^{[2k-1]} : k \in \mathbb{N}\}$. The continuity of T shows that $T|_{E_a}$ is linear. □

Our next technical result establishes that every (not necessarily linear) 2-local triple homomorphism between JB^* -triples is additive on every couple of orthogonal tripotents. The result is a generalization of [14, Lemma 2.2] to the setting of JB^* -triples; it should be noted that, in this more general setting, we need new and independent geometric arguments.

Lemma 2.3. *Let $T : E \rightarrow F$ be a (not necessarily linear) 2-local triple homomorphism between JB^* -triples. Let e and f be two orthogonal tripotents in E . Then $T(e + f) = T(e) + T(f)$.*

Proof. Take a real number $\lambda \in (0, 1]$. In this case we have

$$T(e + \lambda f) = \Phi_{e+\lambda f, e}(e + \lambda f) = T(e) + \lambda \Phi_{e+\lambda f, e}(f)$$

with $\Phi_{e+\lambda f, e}(f) \perp T(e)$. We similarly have

$$T(e + \lambda f) = \Phi_{e+\lambda f, f}(e + \lambda f) = \Phi_{e+\lambda f, f}(e) + \lambda T(f).$$

Combining the above identities we have that

$$\begin{aligned} \Phi_{e+\lambda f, f}(e) &= T(e) + \lambda(\Phi_{e+\lambda f, e}(f) - T(f)) \\ &= T(e) + P_0(T(e))(T(e + \lambda f)) - \lambda T(f). \end{aligned}$$

Since $\Phi_{e+\lambda f, f}(e)$ and $T(e)$ are tripotents and

$$T(e) \perp P_0(T(e))(T(e + \lambda f)) - \lambda T(f),$$

it follows that $P_0(T(e))(T(e + \lambda f)) - \lambda T(f)$ also is a tripotent for every $\lambda \in (0, 1]$.

Clearly, the function $f : [0, 1] \rightarrow \{0, 1\}$ defined by

$$f(\lambda) := \|P_0(T(e))T(e + \lambda f) - \lambda T(f)\|,$$

is continuous with $f(0) = 0$, thus $f(\lambda) = 0 \ \forall \lambda \in [0, 1]$. This implies, in particular, that $f(1) = 0$, or equivalently, $P_0(T(e))T(e + f) = T(f)$, which finishes the proof. \square

The linearity of every (not necessarily linear) 2-local triple homomorphism on finite linear combinations of mutually orthogonal tripotents follows next.

Lemma 2.4. *Let $T : E \rightarrow F$ be a (not necessarily linear) 2-local triple homomorphism between JB^* -triples. Let e_1, \dots, e_n be mutually orthogonal tripotents in E . Then*

- (a) $T\left(\sum_{i=1}^n e_i\right) = \sum_{i=1}^n T(e_i);$
- (b) $T\left(\sum_{i=1}^n \lambda_i e_i\right) = \sum_{i=1}^n \lambda_i T(e_i),$ for every $\lambda_1, \dots, \lambda_n \in \mathbb{C}.$

Proof. (a) We shall argue by induction on n . The case $n = 1$ is clear, while the case $n = 2$ is established in Lemma 2.3. Let us suppose that e_1, \dots, e_n, e_{n+1} are mutually orthogonal tripotents in E . Since $e = e_1 + \dots + e_n$ and e_{n+1} are orthogonal tripotents in E , Lemma 2.3 and the induction hypothesis prove that

$$T\left(\sum_{i=1}^{n+1} e_i\right) = T(e + e_{n+1}) = T(e) + T(e_{n+1}) = \sum_{i=1}^n T(e_i) + T(e_{n+1}).$$

- (b) Fix $j \in \{1, \dots, n\}$ and set $z = \sum_{i=1}^n \lambda_i e_i$ and $e = \sum_{i=1}^n e_i$. The identity

$$\left\{T(e_j), T(e_j), T\left(\sum_{i=1}^n \lambda_i e_i\right)\right\} = \left\{\Phi_{z, e_j}(e_j), \Phi_{z, e_j}(e_j), \Phi_{z, e_j}\left(\sum_{i=1}^n \lambda_i e_i\right)\right\}$$

$$= \Phi_{z,e_j} \left(\left\{ e_j, e_j, \sum_{i=1}^n \lambda_i e_i \right\} \right) = \Phi_{z,e_j}(\lambda_j e_j) = \lambda_j T(e_j),$$

implies that

$$\begin{aligned} T \left(\sum_{i=1}^n \lambda_i e_i \right) &= \Phi_{z,e}(z) = \Phi_{z,e}(\{e, e, z\}) = \{\Phi_{z,e}(e), \Phi_{z,e}(e), \Phi_{z,e}(z)\} \\ &= \{T(e), T(e), T(z)\} = \left\{ T \left(\sum_{j=1}^n e_j \right), T \left(\sum_{j=1}^n e_j \right), T \left(\sum_{i=1}^n \lambda_i e_i \right) \right\} \\ &= (\text{by (a)}) = \left\{ \sum_{j=1}^n T(e_j), \sum_{j=1}^n T(e_j), T \left(\sum_{i=1}^n \lambda_i e_i \right) \right\} = (\text{by orthogonality}) \\ &= \sum_{j=1}^n \left\{ T(e_j), T(e_j), T \left(\sum_{i=1}^n \lambda_i e_i \right) \right\} = \sum_{j=1}^n \lambda_j T(e_j). \end{aligned}$$

□

Let E and F be JB*-triples. We recall that a (not necessarily linear) mapping $f : E \rightarrow F$ is called *orthogonally additive* if $f(a+b) = f(a) + f(b)$ for every $a \perp b$ in E .

Proposition 2.5. *Let E be a JBW*-triple, F a JB*-triple, and suppose that $T : E \rightarrow F$ is a (not necessarily linear) 2-local triple homomorphism. Then T is orthogonally additive.*

Proof. Let a and b be two orthogonal elements in E . The range tripotents $r(a)$ and $r(b)$ are orthogonal, and the JBW*-subtriples $E_2(r(a))$ and $E_2(r(b))$ are also orthogonal (cf. [11, Lemma 1.1]).

For each $\varepsilon > 0$, there exists two algebraic elements $a_\varepsilon = \sum_{k=1}^{m_1} \lambda_k e_k$ and $b_\varepsilon = \sum_{j=1}^{m_2} \mu_j v_j$, where $\lambda_k, \mu_j \in \mathbb{R}$, e_1, \dots, e_{m_1} and v_1, \dots, v_{m_2} are tripotents in $E_2(r(a))$ and $E_2(r(b))$, respectively, and $e_j \perp e_k$, $v_j \perp v_k$ for every $j \neq k$, such that $\|a - a_\varepsilon\| < \frac{\varepsilon}{4}$, and $\|b - b_\varepsilon\| < \frac{\varepsilon}{4}$ (cf. [25, lemma 3.11]). It is clear that $a_\varepsilon + b_\varepsilon$ is a linear combination of mutually orthogonal tripotents. Then, by Lemma 2.1(e) and Lemma 2.4(b),

$$\begin{aligned} \|T(a+b) - T(a) - T(b)\| &= \|T(a+b) - T(a_\varepsilon + b_\varepsilon) + T(a_\varepsilon) + T(b_\varepsilon) - T(a) - T(b)\| \\ &\leq \|T(a+b) - T(a_\varepsilon + b_\varepsilon)\| + \|T(a_\varepsilon) - T(a)\| \\ &\quad + \|T(b_\varepsilon) - T(b)\| < \|(a+b) - (a_\varepsilon + b_\varepsilon)\| + \|a_\varepsilon - a\| + \|b_\varepsilon - b\| < \varepsilon. \end{aligned}$$

Since ε was arbitrarily chosen, we get $T(a+b) = T(a) + T(b)$. □

A simple induction argument, combined with Proposition 2.5, shows:

Corollary 2.6. *Let $(E_i)_{i=1}^n$ be a finite family of JBW^* -triples and let F be a JB^* -triple. Suppose that, for every i , every (not necessarily linear) 2-local triple homomorphism $T : E_i \rightarrow F$ is linear. Then every (not necessarily linear) 2-local triple homomorphism $T : \bigoplus_{i=1, \dots, n}^{\ell_\infty} E_i \rightarrow F$ is linear. \square*

We recall now a result, due to Friedmann and Russo, which has been borrowed from [22, Lemma 1.6].

Lemma 2.7. [22, Lemma 1.6] *Let e be a tripotent in a JB^* -triple E . Then, for each norm-one element $x \in E$ satisfying $P_2(e)x = e$, we have $P_1(e)x = 0$. \square*

In order to make the results more accessible, we have splitted the technical arguments needed in the proofs of our main results into a series of lemmas and propositions, which assure certain almost-linearity properties of 2-local triple homomorphisms.

Lemma 2.8. *Let $T : E \rightarrow F$ be a (not necessarily linear) 2-local triple homomorphism between JB^* -triples. Suppose e is a tripotent in E and $z \in E_0(e)$ with $\|z\| < 1$. Then, for each $w \in E$ and each triple homomorphism $\Phi_{w,e+z} : E \rightarrow F$ satisfying $\Phi_{w,e+z}(w) = T(w)$ and $\Phi_{w,e+z}(e+z) = T(e+z)$, we have $\Phi_{w,e+z}(e) = T(e)$, and consequently, $\Phi_{w,e+z}(E_j(e)) \subseteq F_j(T(e))$, for every $j = 0, 1, 2$.*

Proof. By considering the triple homomorphism $\Phi_{e,e+z}$, we obtain that

$$T(e+z) = \Phi_{e,e+z}(e+z) = T(e) + \Phi_{e,e+z}(z),$$

where $\Phi_{e,e+z}(z) \in F_0(\Phi_{e,e+z}(e)) = F_0(T(e))$ and $\|\Phi_{e,e+z}(z)\| \leq \|z\| < 1$. Since $\|z\| < 1$, $\|\Phi_{e,e+z}(z)\| < 1$ and $T(z) \in F_0(T(e))$ (cf. Lemma 2.1), we have,

$$\begin{aligned} \Phi_{w,e+z}(e) &= \Phi_{w,e+z} \left(\lim_{n \rightarrow \infty} (e+z)^{[3^n]} \right) = \lim_{n \rightarrow \infty} (\Phi_{w,e+z}(e+z))^{[3^n]} \\ &= \lim_{n \rightarrow \infty} (T(e+z))^{[3^n]} = \lim_{n \rightarrow \infty} (T(e) + \Phi_{e,e+z}(z))^{[3^n]} = T(e), \end{aligned}$$

where all the above limits are in the norm-topology. \square

Lemma 2.9. *Let $T : E \rightarrow F$ be a (not necessarily linear) 2-local triple homomorphism between JB^* -triples. Then the following statements hold:*

(a) *For each tripotent e in E , and each $y \in E_1(e)$, we have*

$$T(e+y) = T(e) + T(y);$$

(b) *Suppose e_1, e_2 , and g are tripotents in E satisfying $e_1 \perp e_2$, $e_1, e_2 \in E_2(g)$, $g \in E_1(e_1) \cap E_1(e_2)$. Then, the identity*

$$T(\lambda_1 e_1 + \mu g + \lambda_2 e_2) = \lambda_1 T(e_1) + \mu T(g) + \lambda_2 T(e_2),$$

holds for every $\lambda_1, \lambda_2, \mu \in \mathbb{C}$.

Proof. (a) Let e be a tripotent in E , and let $y \in E_1(e)$. By Lemma 2.1(g), the desired statement is clear when $T(e) = 0$, so we assume that $T(e) \neq 0$. In this case,

$$T(e + y) = \Phi_{e+y,e}(e + y) = T(e) + \Phi_{e+y,e}(y),$$

where $\Phi_{e+y,e}(y) \in F_1(\Phi_{e+y,e}(e)) = F_1(T(e))$, and we also have

$$T(e + y) = \Phi_{e+y,y}(e + y) = \Phi_{e+y,y}(e) + T(y),$$

with $\Phi_{e+y,y}(e)$ being a tripotent. Therefore,

$$\Phi_{e+y,y}(e) = T(e) + \Phi_{e+y,e}(y) - T(y),$$

with $\Phi_{e+y,e}(y) - T(y) \in F_1(T(e))$. It follows from Lemma 2.7 that

$$0 = P_1(T(e))(\Phi_{e+y,y}(e)) = \Phi_{e+y,e}(y) - T(y),$$

witnessing the desired statement.

(b) We can assume that $\lambda_1, \lambda_2, \mu \neq 0$, otherwise the statement is clear from (a) or from Lemma 2.4. To simplify notation, we set $z = \lambda_1 e_1 + \mu g + \lambda_2 e_2$. Applying (a) we get

$$T(z) = \Phi_{z,\lambda_1 e_1 + \mu g}(z) = \lambda_1 T(e_1) + \mu T(g) + \lambda_2 \Phi_{z,\lambda_1 e_1 + \mu g}(e_2).$$

We also have

$$(2.1) \quad T(z) = \Phi_{z,e_2}(z) = \lambda_1 \Phi_{z,e_2}(e_1) + \mu \Phi_{z,e_2}(g) + \lambda_2 T(e_2).$$

Combining these two equalities we have

$$\Phi_{z,\lambda_1 e_1 + \mu g}(e_2) = T(e_2) + \frac{\mu}{\lambda_2}(\Phi_{z,e_2}(g) - T(g)) + \frac{\lambda_1}{\lambda_2}(\Phi_{z,e_2}(e_1) - T(e_1)),$$

where $\Phi_{z,e_2}(e_1) \in F_0(\Phi_{z,e_2}(e_2)) = F_0(T(e_2))$, $T(e_1) \in F_0(T(e_2))$, $\Phi_{z,e_2}(g) \in F_1(\Phi_{z,e_2}(e_2)) = F_1(T(e_2))$, and $T(g) \in F_1(T(e_2))$ (cf. Lemma 2.1(g)). Lemma 2.7 implies that $T(g) = \Phi_{z,e_2}(g)$, and hence (2.1) writes in the form

$$T(z) = \Phi_{z,e_2}(z) = \lambda_1 \Phi_{z,e_2}(e_1) + \mu T(g) + \lambda_2 T(e_2).$$

The last identity implies that $P_2(T(e_2))T(z) = \lambda_2 T(e_2)$, $P_1(T(e_2))T(z) = \mu T(g)$, and $P_0(T(e_2))T(z) = \lambda_1 \Phi_{z,e_2}(e_1)$.

The identity

$$T(z) = \Phi_{z,e_1}(z) = \lambda_1 T(e_1) + \mu \Phi_{z,e_1}(g) + \lambda_2 \Phi_{z,e_1}(e_2),$$

shows that $P_2(T(e_1))T(z) = \lambda_1 T(e_1)$.

Finally, having in mind that $T(z) \in F_2(T(e_1 + e_2)) = F_2(T(e_1) + T(e_2))$, and $F_0(T(e_2)) \cap F_2(T(e_1 + e_2)) = F_2(T(e_1))$ (cf. [25, 1.12]), we have $\Phi_{z,e_2}(e_1) = T(e_1)$. \square

Lemma 2.10. *Let $T : E \rightarrow F$ be a (not necessarily linear) 2-local triple homomorphism from a JBW^* -triple into a JB^* -triple, and let e be a tripotent in E . Then the following statements hold:*

- (a) $T(e + y + z) = T(e) + T(y) + T(z)$, for every $y \in E_1(e)$, and every $z \in E_0(e)$ with $\|z\| < 1$;

- (b) $T(y + z) = T(y) + T(z)$, for every $y \in E_1(e)$, and every $z \in E_0(e)$;
(c) $T(e + y + z) = T(e) + T(y) + T(z)$, for every $y \in E_1(e)$, and every $z \in E_0(e)$. Consequently, $T(\lambda e + y + z) = \lambda T(e) + T(y) + T(z)$, for every $y \in E_1(e)$, every $z \in E_0(e)$, and every $\lambda \in \mathbb{C}$.

Proof. Throughout the proof we set $w = e + y + z$.

(a) We assume first that $T(e) = 0$. By Lemma 2.1(g), $T(y) = 0$. By Lemma 2.8, $\Phi_{w,e+z}(e) = T(e) = 0$ and hence $\Phi_{w,e+z}(y) \in F_1(\Phi_{w,e+z}(e)) = \{0\}$. If we write

$$\begin{aligned} T(e + y + z) &= T(w) = \Phi_{w,e+z}(w) = T(e + z) + \Phi_{w,e+z}(y) \\ &= T(e + z) = (\text{by Proposition 2.5}) = T(e) + T(z) = 0. \end{aligned}$$

Suppose now that $T(e) \neq 0$. Proposition 2.5 implies that

$$T(w) = \Phi_{w,e+z}(w) = T(e + z) + \Phi_{w,e+z}(y) = T(e) + T(z) + \Phi_{w,e+z}(y).$$

By Lemma 2.8, $\Phi_{w,e+z}(e) = T(e)$, and in particular $\Phi_{w,e+z}(y) \in F_1(T(e))$. We also have

$$T(w) = \Phi_{w,y}(w) = T(y) + \Phi_{w,y}(e + z),$$

and hence

$$\Phi_{w,y}(e + z) = T(e) + \Phi_{w,e+z}(y) - T(y) + T(z).$$

Having in mind that $\|\Phi_{w,y}(e + z)\| \leq 1$, Lemma 2.7 implies that

$$0 = P_1(T(e))\Phi_{w,y}(e + z) = \Phi_{w,e+z}(y) - T(y).$$

(b) Since T is 1-homogeneous, we may assume without loss of generality that $\|z\| < 1$. As in the previous case, let us assume that $T(e) = 0$. Under these assumptions, Lemma 2.8 implies that $\Phi_{y+z,e+z}(e + y + z - e)(e) = T(e) = 0$ and hence $\Phi_{y+z,e+z}(E_2(e) \oplus E_1(e)) = \{0\}$. Then

$$\begin{aligned} T(y + z) &= \Phi_{y+z,e+z}(e + y + z - e) = T(e + z) + \Phi_{y+z,e+z}(y - e) \\ &= T(e + z) = (\text{by Proposition 2.5}) = T(e) + T(z) = T(z). \end{aligned}$$

We consider now the case $T(e) \neq 0$. Since we are assuming $\|z\| < 1$, it follows from (a) that

$$\begin{aligned} T(y + z) &= \Phi_{y+z,w}(e + y + z - e) = \Phi_{y+z,w}(e + y + z) - \Phi_{y+z,w}(e) \\ &= T(e) + T(y) + T(z) - \Phi_{y+z,w}(e). \end{aligned}$$

Considering that $T(y + z) \in F_1(T(e)) + F_0(T(e))$ (see Lemma 2.1(f)) we have $P_2(T(e))\Phi_{y+z,w}(e) = T(e)$. Lemma 2.7, applied in the identity $\Phi_{y+z,w}(e) = T(e) + T(y) - T(y + z) + T(z)$, shows that $P_1(T(e))T(y + z) = T(y)$.

By Corollary 2.5, we get

$$\begin{aligned} T(y + z) &= \Phi_{y+z,e+z}(e + z) + \Phi_{y+z,e+z}(y - e) = T(e + z) + \Phi_{y+z,e+z}(y - e) \\ &= T(e) + T(z) + \Phi_{y+z,e+z}(y) - \Phi_{y+z,e+z}(e). \end{aligned}$$

By assumptions $\|z\| < 1$. Thus, applying Lemma 2.8 we show $\Phi_{y+z, e+z}(e) = T(e)$. Therefore, $\Phi_{y+z, e+z}(y) \in F_1(T(e))$ and $P_0(T(e))T(y+z) = T(z)$, which proves that

$$T(y+z) = P_1(T(e))T(y+z) + P_0(T(e))T(y+z) = T(y) + T(z).$$

(c) We begin with the case $T(e) \neq 0$. For each real number $\lambda \in [0, 1]$, we denote $w_\lambda := e + y + \lambda z$. By the assumptions on T

$$T(w_\lambda) = \Phi_{w_\lambda, e}(w_\lambda) = T(e) + \Phi_{w_\lambda, e}(y) + \lambda \Phi_{w_\lambda, e}(z),$$

where $\Phi_{w_\lambda, e}(y) \in F_1(T(e))$, and $\Phi_{w_\lambda, e}(z) \in F_0(T(e))$. Applying (b) we deduce that

$$T(w_\lambda) = \Phi_{w_\lambda, y+\lambda z}(w_\lambda) = T(y) + \lambda T(z) + \Phi_{w_\lambda, y+\lambda z}(e).$$

The above identities show that

$$\Phi_{w_\lambda, y+\lambda z}(e) = T(e) + (\Phi_{w_\lambda, e}(y) - T(y)) + \lambda(\Phi_{w_\lambda, e}(z) - T(z)),$$

and Lemma 2.7 applies to assure that $\Phi_{w_\lambda, e}(y) = T(y)$. Therefore,

$$\Phi_{w_\lambda, y+\lambda z}(e) = T(e) + \lambda(\Phi_{w_\lambda, e}(z) - T(z)).$$

Since $\Phi_{w_\lambda, y+\lambda z}(e)$ is a tripotent, we deduce that

$$P_0(T(e))\Phi_{w_\lambda, y+\lambda z}(e) = \lambda(\Phi_{w_\lambda, e}(z) - T(z)) = P_0(T(e))(T(w_\lambda)) - \lambda T(z)$$

is a tripotent.

Finally the mapping $f : [0, 1] \rightarrow \{0, 1\}$ given by

$$f(\lambda) = \|P_0(T(e))(T(w_\lambda)) - \lambda T(z)\|$$

is (norm) continuous and $f(0) = 0$, then $f(\lambda) = 0$ for every $\lambda \in [0, 1]$, and hence $\Phi_{w_\lambda, e}(z) = T(z)$, which gives the desired statement.

Suppose, finally, that $T(e) = 0$. Lemma 2.1(g) implies that $T(y) = 0$. Let us observe that $\Phi_{w_\lambda, e}(e) = T(e) = 0$. The identities

$$T(w_\lambda) = \Phi_{w_\lambda, e}(w_\lambda) = T(e) + \Phi_{w_\lambda, e}(y) + \lambda \Phi_{w_\lambda, e}(z) = \lambda \Phi_{w_\lambda, e}(z),$$

$T(w_\lambda) = \Phi_{w_\lambda, y+\lambda z}(w_\lambda) = \Phi_{w_\lambda, y+\lambda z}(e) + T(y) + \lambda T(z) = \Phi_{w_\lambda, y+\lambda z}(e) + \lambda T(z)$, show that

$$\Phi_{w_\lambda, y+\lambda z}(e) = \lambda \Phi_{w_\lambda, e}(z) - \lambda T(z) = T(w_\lambda) - \lambda T(z),$$

for every $\lambda \in [0, 1]$. Since, for every $0 \leq \lambda \leq 1$, $\Phi_{w_\lambda, y+\lambda z}(e)$ is a tripotent, the function $f : [0, 1] \rightarrow \mathbb{R}$, $f(\lambda) := \|\Phi_{w_\lambda, y+\lambda z}(e)\| = \|T(w_\lambda) - \lambda T(z)\|$ is continuous and takes only the values 0 and 1. Since $f(0) = 0$, we conclude that $f(\lambda) = 0$, for every $\lambda \in [0, 1]$, which proves $T(e+y+z) - T(z) = 0$. \square

We recall that, given a conjugation (conjugate linear isometry of period 2), σ , on a complex Hilbert space H with $\dim(H) = n \in \mathbb{N} \cup \{\infty\}$, the mapping $x \mapsto x^t := \sigma x^* \sigma$ defines a linear involution on $L(H)$. The type-3 Cartan factor, denoted by III_n , is the subtriple of $L(H)$ formed by the t -symmetric operators. Following standard notation, $S_2(\mathbb{C})$ will denote III_2 .

Corollary 2.11. *Let F be a JB^* -triple and let $T : C \rightarrow F$ be a (not necessarily linear) 2-local triple-homomorphism, where C is $M_2(\mathbb{C})$ or $S_2(\mathbb{C})$. Then T is linear and a triple homomorphism.*

Proof. Suppose first that $C = M_2(\mathbb{C})$. We set $e_1 = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$, $e_2 = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$, $y_1 = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$, and $y_2 = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$. Lemma 2.10(c) implies that

$$\begin{aligned} T(\lambda_1 e_1 + \mu_1 y_1 + \mu_2 y_2 + \lambda_2 e_2) &= \lambda_1 T(e_1) + T(\mu_1 y_1 + \mu_2 y_2) + \lambda_2 T(e_2) \\ &= (\text{Proposition 2.5 applied to } y_1 \perp y_2) = \\ &= \lambda_1 T(e_1) + \mu_1 T(y_1) + \mu_2 T(y_2) + \lambda_2 T(e_2). \end{aligned}$$

The linearity follows from the fact that $\{e_1, y_1, y_2, e_2\}$ is a basis of $M_2(\mathbb{C})$.

For the statement concerning $S_2(\mathbb{C})$, we observe that we can assume that $\sigma : H = \ell_2^2 \rightarrow H = \ell_2^2$ is the mapping given by $\sigma(t_1, t_2) = (\bar{t}_1, \bar{t}_2)$. Considering $e_1 = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$, $e_2 = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$, and $y = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$, Lemma 2.10(c) implies that

$$T(\lambda_1 e_1 + \mu y + \lambda_2 e_2) = \lambda_1 T(e_1) + \mu T(y) + \lambda_2 T(e_2),$$

which proves that T is linear. \square

Proposition 2.12. *Let $T : E \rightarrow F$ be a (not necessarily linear) 2-local triple homomorphism from a JBW^* -triple into a JB^* -triple, and let e be a tripotent in E . Then $T(x + y) = T(x) + T(y)$, for all $x \in E_2(e)$, $y \in E_1(e)$.*

Proof. Let us observe that by Lemma 2.1(g), we may assume that $T(e) \neq 0$. By the norm density of algebraic elements in $E_2(e)$ (cf. [25, lemma 3.11]), together with the continuity of T , it is enough to prove that, for every algebraic element a in $E_2(e)$ (i.e. $a = \sum_{k=1}^m \lambda_k e_k$, where $\lambda_k \in \mathbb{R}$, and e_1, \dots, e_{m_1} are mutually orthogonal tripotents in $E_2(e)$), we have $T(a + y) = T(a) + T(y)$. We shall prove this statement by induction on the number m of mutually orthogonal tripotents whose linear combination coincides with a .

For the case $m = 1$, we may assume that $a = \lambda_1 e_1 \in E_2(e)$ with $\lambda_1 \neq 0$. Since $y \in E_1(e)$, it follows from Peirce rules that y writes in the form $y = y_1 + y_0$, where $y_k = P_k(e_1)y$, $k = 1, 0$. By Lemma 2.10(c),

$$T(a + y) = T(\lambda_1 e_1 + y_1 + y_0) = T(\lambda_1 e_1) + T(y_1) + T(y_0),$$

and by Lemma 2.10(b), $T(y_1) + T(y_0) = T(y_1 + y_0) = T(y)$, then $T(a + y) = T(a) + T(y)$.

Suppose, by the induction hypothesis, that for every algebraic element b in $E_2(e)$ which is a linear combination of m mutually orthogonal tripotents in $E_2(e)$, we have

$$T(b + y) = T(b) + T(y),$$

for every $y \in E_1(e)$. Let $a = \sum_{i=1}^{m+1} \lambda_i e_i$ be an algebraic element in $E_2(e)$,

and denote by f the tripotent $\sum_{i=1}^{m+1} e_i$. Applying the Peirce decompositions of $a + y$ associated with f and e_1 , we have $a + y = a + y_1 + y_0$, where $y_1 = P_1(f)y$ and $y_0 = P_0(f)y$, and

$$a + y = \lambda_1 e_1 + P_1(e_1)y_1 + \left(\sum_{i=2}^{m+1} \lambda_i e_i + P_0(e_1)y_1 + y_0 \right),$$

where $\left(\sum_{i=2}^{m+1} \lambda_i e_i + P_0(e_1)y_1 + y_0 \right) \in E_0(e_1)$.

Lemma 2.10(c) implies that

$$(2.2) \quad T(a + y) = T(\lambda_1 e_1) + T(P_1(e_1)y_1) + T\left(\sum_{i=2}^{m+1} \lambda_i e_i + P_0(e_1)y_1 + y_0\right).$$

We observe that $e_1, f \in E_2(e)$, therefore $P_j(e_1)P_k(e) = P_k(e)P_j(e_1)$ and $P_j(f)P_k(e) = P_k(e)P_j(f)$, for every $j, k \in \{0, 1, 2\}$ (cf. [22, Lemma 1.10]).

The induction hypothesis, applied to $\sum_{i=2}^{m+1} \lambda_i e_i \in E_2(e)$ and $P_0(e_1)y_1 + y_0 = P_0(e_1)P_1(f)(y) + P_0(f)(y) = P_0(e_1)P_1(f)P_1(e)(y) + P_0(f)P_1(e)(y) = P_1(e)(P_0(e_1)P_1(f)(y) + P_0(f)(y)) \in E_1(e)$, assures that

$$(2.3) \quad T\left(\sum_{i=2}^{m+1} \lambda_i e_i + P_0(e_1)y_1 + y_0\right) = T\left(\sum_{i=2}^{m+1} \lambda_i e_i\right) + T(P_0(e_1)y_1 + y_0).$$

Finally, by Lemma 2.4

$$(2.4) \quad T(\lambda_1 e_1) + T\left(\sum_{i=2}^{m+1} \lambda_i e_i\right) = T(a).$$

Since $P_1(e_1)y_1 \in E_1(e_1)$, $P_0(e_1)y_1 + y_0 \in E_0(e_1)$, Lemma 2.10 (b) implies that

$$\begin{aligned} T(P_1(e_1)y_1) + T(P_0(e_1)y_1 + y_0) &= T(P_1(e_1)y_1 + P_0(e_1)y_1 + y_0) \\ &= T(y_1 + y_0) = T(y), \end{aligned}$$

which combined with (2.2), (2.3), and (2.4) prove $T(a+y) = T(a)+T(y)$. \square

Our series of technical results on 2-local triple homomorphisms concludes with an strengthened version of Lemma 2.10.

Lemma 2.13. *Let $T : E \rightarrow F$ be a (not necessarily linear) 2-local triple homomorphism from a JBW^* -triple into a JB^* -triple, and let e be a tripotent in E . Then the following statements hold:*

- (a) $T(e + ih + y + z) = T(e) + iT(h) + T(y) + T(z)$, for every $h \in E_2(e)_{sa}$, $y \in E_1(e)$, and every $z \in E_0(e)$ with $\|z\| < 1$;
- (b) $T(ih + y + z) = iT(h) + T(y) + T(z)$, for every $h \in E_2(e)_{sa}$, $y \in E_1(e)$, and every $z \in E_0(e)$;
- (c) $T(e + ih + y + z) = T(e) + iT(h) + T(y) + T(z)$, for every $h \in E_2(e)_{sa}$, $y \in E_1(e)$, and every $z \in E_0(e)$. Consequently,

$$T(\lambda e + ih + y + z) = \lambda T(e) + iT(h) + T(y) + T(z),$$

for every $\lambda \in \mathbb{C}$, $h \in E_2(e)_{sa}$, $y \in E_1(e)$, and every $z \in E_0(e)$.

Proof. Along this proof we set $w = e + ih + y + z$.

We shall assume first that $T(e) = 0$. By Lemma 2.1(g), we have $T(ih) = T(y) = 0$.

- (a) It follows from Lemma 2.8 that $\Phi_{w,e+z}(e) = T(e) = 0$, and hence $\Phi_{w,e+z}(E_2(e) \oplus E_1(e)) = \{0\}$. Therefore,

$$\begin{aligned} T(e + ih + y + z) &= \Phi_{w,e+z}(w) = T(e + z) + i\Phi_{w,e+z}(h) + \Phi_{w,e+z}(y) \\ &= (\text{by Proposition 2.5}) = T(e) + T(z) + i\Phi_{w,e+z}(h) + \Phi_{w,e+z}(y) = T(z). \end{aligned}$$

- (b) Since T is 1-homogeneous, we may assume, without losing generality, that $\|z\| < 1$. Lemma 2.8 implies that $\Phi_{w-e,e+z}(e) = T(e) = 0$. We write

$$\begin{aligned} T(ih + y + z) &= \Phi_{w-e,e+z}(w - e) = T(e + z) + i\Phi_{w-e,e+z}(h) + \Phi_{w-e,e+z}(y) \\ &= T(e + z) = (\text{by Proposition 2.5}) = T(e) + T(z) = T(z). \end{aligned}$$

- (c) Given $\lambda \in [0, 1]$, we set $w_\lambda := e + ih + y + \lambda z$. The identities:

$$T(w_\lambda) = \Phi_{w_\lambda,e}(w_\lambda) = T(e) + i\Phi_{w_\lambda,e}(h) + \Phi_{w_\lambda,e}(y) + \lambda\Phi_{w_\lambda,e}(z) = \lambda\Phi_{w_\lambda,e}(z),$$

and

$$\begin{aligned} T(w_\lambda) &= \Phi_{w_\lambda,ih+y+\lambda z}(w_\lambda) = \Phi_{w_\lambda,ih+y+\lambda z}(e) + T(ih + y + \lambda z) \\ &= (\text{by (b)}) = \Phi_{w_\lambda,ih+y+\lambda z}(e) + iT(h) + T(y) + \lambda T(z) = \Phi_{w_\lambda,ih+y+\lambda z}(e) + \lambda T(z) \end{aligned}$$

assure that

$$\Phi_{w_\lambda,ih+y+\lambda z}(e) = T(w_\lambda) - \lambda T(z).$$

Arguing as in the proof of Lemma 2.10(c) (case $T(e) = 0$), we deduce that $T(w_\lambda) = \lambda T(z)$, for every $0 \leq \lambda \leq 1$.

We suppose from this moment that $T(e) \neq 0$.

- (a) We begin with the identity

$$\begin{aligned} T(w) &= \Phi_{w,e+z}(w) = T(e + z) + i\Phi_{w,e+z}(h) + \Phi_{w,e+z}(y) \\ &= (\text{by Proposition 2.5}) = T(e) + T(z) + i\Phi_{w,e+z}(h) + \Phi_{w,e+z}(y). \end{aligned}$$

We deduce, by Lemma 2.8, that $\Phi_{w,e+z}(e) = T(e)$, which, in particular, gives $\Phi_{w,e+z}(y) \in F_1(T(e))$ and $\Phi_{w,e+z}(h) \in F_2(T(e))_{sa}$.

On the other hand,

$$\begin{aligned} T(w) &= \Phi_{w,ih+y}(w) = T(ih+y) + \Phi_{w,ih+y}(e+z) \\ &= (\text{by Proposition 2.12}) = iT(h) + T(y) + \Phi_{w,ih+y}(e+z), \end{aligned}$$

and hence

$$\Phi_{w,ih+y}(e+z) = T(e) + i(\Phi_{w,e+z}(h) - T(h)) + (\Phi_{w,e+z}(y) - T(y)) + T(z).$$

The element $T(e) + i(\Phi_{w,e+z}(h) - T(h))$ lies in the JB*-algebra $F_2(T(e))$ and $(\Phi_{w,e+z}(h) - T(h)) \in F_2(T(e))_{sa}$ (cf. Lemma 2.1(f)) with

$$\|T(e) + i(\Phi_{w,e+z}(h) - T(h))\| \leq \|\Phi_{w,ih+y}(e+z)\| \leq 1.$$

Therefore

$$1 \geq \|T(e) + i(\Phi_{w,e+z}(h) - T(h))\|^2 \geq 1 + \|\Phi_{w,e+z}(h) - T(h)\|^2,$$

witnessing that $\Phi_{w,e+z}(h) = T(h)$. Having in mind that $\|\Phi_{w,ih+y}(e+z)\| \leq 1$, and

$$\Phi_{w,ih+y}(e+z) = T(e) + (\Phi_{w,e+z}(y) - T(y)) + T(z),$$

Lemma 2.7 implies that

$$0 = P_1(T(e))\Phi_{w,ih+y}(e+z) = \Phi_{w,e+z}(y) - T(y).$$

(b) Since T is 1-homogeneous, we may assume, without loss of generality, that $\|z\| < 1$. Denoting $a = ih + y + z$, it follows that

$$\begin{aligned} T(ih + y + z) &= \Phi_{a,w}(e + ih + y + z - e) = \Phi_{a,w}(e + ih + y + z) - \Phi_{a,w}(e) \\ &= T(e + ih + y + z) - \Phi_{a,w}(e) = (\text{by (a)}) = T(e) + iT(h) + T(y) + T(z) - \Phi_{a,w}(e). \end{aligned}$$

On the other hand,

$$\begin{aligned} T(ih + y + z) &= \Phi_{a,e+z}(e + ih + y + z - e) \\ &= T(e + z) + i\Phi_{a,e+z}(h) + \Phi_{a,e+z}(y) - \Phi_{a,e+z}(e) = (\text{by Proposition 2.5}) \\ &= T(e) + T(z) + i\Phi_{a,e+z}(h) + \Phi_{a,e+z}(y) - \Phi_{a,e+z}(e). \end{aligned}$$

We conclude from Lemma 2.8 that $\Phi_{a,e+z}(e) = T(e)$. Therefore $\Phi_{a,e+z}(h) \in F_2(T(e))_{sa}$, $\Phi_{a,e+z}(y) \in F_1(T(e))$,

$$T(ih + y + z) = i\Phi_{a,e+z}(h) + \Phi_{a,e+z}(y) + T(z),$$

and hence

$$\Phi_{a,w}(e) = T(e) + i(T(h) - \Phi_{a,e+z}(h)) + (T(y) - \Phi_{a,e+z}(y)).$$

The arguments given in the final part of the proof of (a) show that $T(h) = \Phi_{a,e+z}(h)$ and $T(y) = \Phi_{a,e+z}(y)$.

(c) For each real number $\lambda \in [0, 1]$, we denote $w_\lambda := e + ih + y + \lambda z$. By the assumptions

$$T(w_\lambda) = \Phi_{w_\lambda,e}(w_\lambda) = T(e) + i\Phi_{w_\lambda,e}(h) + \Phi_{w_\lambda,e}(y) + \lambda\Phi_{w_\lambda,e}(z),$$

where $\Phi_{w_\lambda,e}(h) \in F_2(T(e))_{sa}$, $\Phi_{w_\lambda,e}(y) \in F_1(T(e))$, and $\Phi_{w_\lambda,e}(z) \in F_0(T(e))$. Applying (b) we deduce that

$$T(w_\lambda) = \Phi_{w_\lambda,ih+y+\lambda z}(w_\lambda) = iT(h) + T(y) + \lambda T(z) + \Phi_{w_\lambda,ih+y+\lambda z}(e).$$

The above identities show that

$$\begin{aligned}\Phi_{w_\lambda, ih+y+\lambda z}(e) &= T(e) + i(\Phi_{w_\lambda, e}(h) - T(h)) + (\Phi_{w_\lambda, e}(y) - T(y)) \\ &\quad + \lambda(\Phi_{w_\lambda, e}(z) - T(z)).\end{aligned}$$

Repeating again the arguments given in the final part of the proof of (a) we obtain $T(h) = \Phi_{w_\lambda, e}(h)$ and $T(y) = \Phi_{w_\lambda, e}(y)$. Therefore,

$$\Phi_{w_\lambda, ih+y+\lambda z}(e) = T(e) + \lambda(\Phi_{w_\lambda, e}(z) - T(z)).$$

Since $\Phi_{w_\lambda, ih+y+\lambda z}(e)$ is a tripotent, we deduce that

$$P_0(T(e))\Phi_{w_\lambda, ih+y+\lambda z}(e) = \lambda(\Phi_{w_\lambda, e}(z) - T(z)) = P_0(T(e))(T(w_\lambda)) - \lambda T(z)$$

is a tripotent.

Finally the mapping $f : [0, 1] \rightarrow \{0, 1\}$ given by

$$f(\lambda) = \|P_0(T(e))(T(w_\lambda)) - \lambda T(z)\|$$

is (norm) continuous and $f(0) = 0$. Then $f(\lambda) = 0$ for every $\lambda \in [0, 1]$, and hence $\Phi_{w_1, e}(z) = T(z)$, which gives the desired statement. \square

3. 2-LOCAL TRIPLE HOMOMORPHISMS ON A JBW^* -ALGEBRA OR ON A JBW^* -TRIPLE

In this section we establish the main results of the paper. Our study on 2-local triple homomorphisms will culminate in a result asserting that every (not necessarily linear) 2-local triple homomorphism from a JBW^* -triple into a JB^* -triple is linear and a triple homomorphism. In a first step we consider 2-local triple homomorphisms whose domains are JBW^* -algebras.

3.1. 2-local triple homomorphisms on a JBW^* -algebra. The aim of this subsection is to study 2-local triple homomorphisms from a JBW^* -algebra or from a von Neumann algebra into a JB^* -triple. The results in these settings are interesting by themselves but also play a crucial role in the proof of our main result for JBW^* -triples.

Let $\Phi : \mathcal{J} \rightarrow F$ be a triple homomorphism from a unital JB^* -algebra into a JB^* -triple. Clearly $\Phi(1)$ is a tripotent in F and $F_2(\Phi(1))$ is a JB^* -algebra with unit $\Phi(1)$. Given a in \mathcal{J} , the identities

$$\{\Phi(1), \Phi(1), \Phi(a)\} = \Phi\{1, 1, a\} = \Phi(a),$$

and

$$\Phi(a)^{\sharp_{\Phi(1)}} = \{\Phi(1), \Phi(a), \Phi(1)\} = \Phi\{1, a, 1\} = \Phi(a^*),$$

prove that $\Phi(\mathcal{J}) \subseteq F_2(\Phi(1))$ and $\Phi(\mathcal{J}_{sa}) \subseteq F_2(\Phi(1))_{sa}$. More precisely, Φ is $F_2(\Phi(1))$ -valued and $\Phi : \mathcal{J} \rightarrow F_2(\Phi(1))$ is a unital Jordan $*$ -homomorphism between unital JB^* -algebras. For 2-local triple homomorphisms we have:

Lemma 3.1. *Let $T : \mathcal{J} \rightarrow F$ be a (not necessarily linear) 2-local triple homomorphism from a unital JB^* -algebra into a JB^* -triple. Then the following statements hold:*

(a) $T(\mathcal{J}) \subseteq F_2(T(1))$;

- (b) $T(\mathcal{J}_{sa}) \subseteq F_2(T(1))_{sa}$;
- (c) $T(a)$ is positive in $F_2(T(1))$ whenever a is positive in \mathcal{J} .

Proof. For each $a \in \mathcal{J}$, and $b \in \mathcal{J}_{sa}$, the comments preceding this lemma assure that $T(a) = \Phi_{1,a}(a) \in F_2(\Phi_{1,a}(1)) = F_2(T(1))$, and $T(b) = \Phi_{1,b}(b) \in F_2(\Phi_{1,b}(1))_{sa} = F_2(T(1))_{sa}$, which proves (a) and (b).

To prove (c), suppose a is a positive element in \mathcal{J} . Since the triple homomorphism $\Phi_{a,1} : \mathcal{J} \rightarrow F_2(\Phi_{a,1}(1)) = F_2(T(1))$ is a unital Jordan *-homomorphism between unital JB*-algebras, $T(a) = \Phi_{1,a}(a)$ is positive in $F_2(\Phi_{a,1}(1)) = F_2(T(1))$. \square

Let \mathcal{J} be a JB*-algebra and let E be a JB*-triple. Following the notation employed in [2] and [10], a *quasi-linear functional* on \mathcal{J} is a function $\rho : \mathcal{J} \rightarrow \mathbb{C}$ such that

- (i) $\rho|_{\mathcal{J}_{<h>}} : \mathcal{J}_{<h>} \rightarrow \mathbb{C}$ is a linear functional for each $h \in \mathcal{J}_{sa}$, where $\mathcal{J}_{<h>}$ denotes the JB*-subalgebra generated by h ;
- (ii) $\rho(a + ib) = \rho(a) + i\rho(b)$, for every $a, b \in \mathcal{J}_{sa}$.

If we also assume that, for each $h \in \mathcal{J}_{sa}$, $\rho|_{\mathcal{J}_{<h>}}$ is a positive linear functional, we shall say that ρ is *positive quasi-linear functional* on \mathcal{J} . A mapping $\rho : E \rightarrow \mathbb{C}$ is said to be a *quasi-linear functional* on E if for every a in E , the restriction of ρ to the JB*-subtriple, E_a , of E generated by a is linear.

Let $T : E \rightarrow F$ be a (not necessarily linear) 2-local triple homomorphism between JB*-triples. For each $\phi \in F^*$, Lemma 2.2 assures that $\phi \circ T : E \rightarrow \mathbb{C}$ is a quasi-linear functional on E in the triple sense. Our next proposition shows that a stronger property holds for 2-local triple homomorphisms from a JBW*-algebra into a JB*-triple.

Proposition 3.2. *Let $T : \mathcal{J} \rightarrow F$ be a (not necessarily linear) 2-local triple homomorphism from a JBW*-algebra into a JB*-triple. Then*

$$T(a + ib) = T(a) + iT(b),$$

for every $a, b \in \mathcal{J}_{sa}$.

Proof. It is known that every $a \in \mathcal{J}_{sa}$, can be approximated in norm by a finite (real) linear combination of mutually orthogonal (non-zero) projections in \mathcal{J} ([24, Proposition 4.2.3]). Since T is continuous, it is enough to prove that

$$(3.1) \quad T(a + ib) = T(a) + iT(b),$$

for every $b \in \mathcal{J}_{sa}$ and $a = \sum_{k=1}^m \lambda_k p_k$, where $\lambda_k \in \mathbb{R} \setminus \{0\}$ and p_1, \dots, p_m are mutually orthogonal projections in \mathcal{J} . We shall prove (3.1) by induction on m .

For the case $m = 1$, we assume that $a = \lambda p$ for a non-zero projection p and $\lambda \in \mathbb{R} \setminus \{0\}$. The element b writes in the form $b = P_2(p)(b) + P_1(p)(b) +$

$P_0(p)(b)$, and since $b = b^*$ and p is a projection, $P_2(p)(b) \in \mathcal{J}_{sa}$. Applying Lemma 2.13(c), we have

$$\begin{aligned} T(a + ib) &= \lambda T(p) + iT(P_2(p)(b)) + iT(P_1(p)(b)) + iT(P_0(p)(b)) \\ &= T(a) + iT(P_2(p)(b)) + iT(P_1(p)(b)) + iT(P_0(p)(b)), \end{aligned}$$

by Lemma 2.13(b),

$$= T(a) + T(iP_2(p)(b) + iP_1(p)(b) + iP_0(p)(b)) = T(a + ib).$$

Suppose, by the induction hypothesis, that (3.1) is true for every algebraic element $\sum_{k=1}^{m_1} \mu_k q_k$, with $m_1 \leq m$, $\lambda_k \in \mathbb{R} \setminus \{0\}$ and q_1, \dots, q_{m_1} mutually orthogonal projections in \mathcal{J} . Let us take an algebraic element of the form $a = \sum_{k=1}^{m+1} \lambda_k p_k$. Let us write $b = P_2(p_1)(b) + P_1(p_1)(b) + P_0(p_1)(b)$, and since $b = b^*$ and p_1 is a projection, $P_2(p_1)(b), P_0(p_1)(b) \in \mathcal{J}_{sa}$. Let us observe that

$$T(a + ib) = T\left(\lambda_1 p_1 + iP_2(p_1)(b) + iP_1(p_1)(b) + \sum_{k=2}^{m+1} \lambda_k p_k + iP_0(p_1)(b)\right),$$

where $P_2(p_1)(b) \in \mathcal{J}_2(p_1)_{sa}$, $iP_1(p_1)(b) \in \mathcal{J}_1(p_1)$, and $\sum_{k=2}^{m+1} \lambda_k p_k + iP_0(p_1)(b)$ lies in $\mathcal{J}_0(p_1)$. Lemma 2.13(c) implies that

$$\begin{aligned} T(a + ib) &= \lambda_1 T(p_1) + iT(P_2(p_1)(b)) + T(iP_1(p_1)(b)) \\ &\quad + T\left(\sum_{k=2}^{m+1} \lambda_k p_k + iP_0(p_1)(b)\right) = (\text{by the induction hypothesis}) = \\ &= \lambda_1 T(p_1) + iT(P_2(p_1)(b)) + T(iP_1(p_1)(b)) + T\left(\sum_{k=2}^{m+1} \lambda_k p_k\right) + T(iP_0(p_1)(b)) \\ &= (\text{by Proposition 2.5}) = T\left(\lambda_1 p_1 + \sum_{k=2}^{m+1} \lambda_k p_k\right) + iT(P_2(p_1)(b)) \\ &\quad + T(iP_1(p_1)(b)) + T(iP_0(p_1)(b)) = (\text{by Lemma 2.13(b) with } e = p_1) \\ &= T(a) + iT(P_2(p_1)(b) + P_1(p_1)(b) + P_0(p_1)(b)) = T(a) + iT(b). \end{aligned}$$

□

We can establish now a generalization of [14, Theorem 4.1] for 2-local triple homomorphisms.

Theorem 3.3. *Let \mathcal{J} be a JBW^* -algebra with no Type I_2 direct summand and let F be a JB^* -triple. Suppose $T : \mathcal{J} \rightarrow F$ is a (not necessarily linear) 2-local triple homomorphism. Then T is linear and a (continuous) triple homomorphism. More concretely, $T : \mathcal{J} \rightarrow F_2(T(1))$ is a linear unital Jordan $*$ -homomorphism between JB^* -algebras.*

Proof. We know that $T(1)$ is a tripotent in F (cf. Lemma 2.1). By Lemma 3.1 $T(\mathcal{J}) \subseteq F_2(T(1))$ and $T(\mathcal{J}_{sa}) \subseteq F_2(T(1))_{sa}$.

Furthermore, given a projection p in \mathcal{J} , since $p \leq 1$, Lemma 2.1(b) and (d) assure that $T(p)$ is a tripotent in $F_2(T(1))$ with $T(p) \leq T(1)$, which implies that $T(p)$ is a projection in $F_2(T(1))$.

Fix an arbitrary norm-one positive functional φ in $F_2(T(1))^*$. Let $\mathcal{P}(\mathcal{J})$ denote the lattice of projections in \mathcal{J} . The mapping

$$\begin{aligned} \mu_\varphi : \mathcal{P}(\mathcal{J}) &\rightarrow \mathbb{R} \\ \mu_\varphi(p) &:= \varphi(T(p)), \end{aligned}$$

is a finitely additive quantum measure on $\mathcal{P}(\mathcal{J})$ in the terminology employed in [9], i.e. $\mu_\varphi(1) = 1$ and $\mu_\varphi(p_1 + \dots + p_m) = \mu_\varphi(p_1) + \dots + \mu_\varphi(p_m)$, whenever p_1, \dots, p_m are mutually orthogonal projections in \mathcal{J} (this statement follows from Lemma 2.4(a) and the fact that $T(1)$ is the unit element in $F_2(T(1))$). By the Bunce-Wright-Mackey-Gleason theorem [9, Theorem 2.1], there exists a positive linear functional $\phi_\varphi \in \mathcal{J}_{sa}^*$ such that

$$\varphi(T(p)) = \mu_\varphi(p) = \phi_\varphi(p),$$

for every $p \in \mathcal{P}(\mathcal{J})$. It follows from Lemma 2.4(b), the continuity of T , φ , and ϕ_φ , and the norm density of algebraic elements in \mathcal{J}_{sa} that

$$\varphi(T(a)) = \phi_\varphi(a),$$

for every $a \in \mathcal{J}_{sa}$. Therefore,

$$\begin{aligned} \varphi(T(a+b)) &= \phi_\varphi(a+b) = \phi_\varphi(a) + \phi_\varphi(b) \\ &= \varphi(T(a)) + \varphi(T(b)) = \varphi(T(a) + T(b)), \end{aligned}$$

for every $a, b \in \mathcal{J}_{sa}$. Since the positive norm-one functionals in $F_2(T(1))^*$ separate the points of $F_2(T(1))_{sa}$ (cf. [24, Lemma 3.6.8]), we deduce that $T(a+b) = T(a) + T(b)$, for every $a, b \in \mathcal{J}_{sa}$, that is, the restricted mapping $T|_{\mathcal{J}_{sa}} : \mathcal{J}_{sa} \rightarrow F_2(T(1))_{sa} \subseteq F$ is linear.

Finally, Proposition 3.2 shows that $T(a+ib) = T(a) + iT(b)$, for every $a, b \in \mathcal{J}_{sa}$, which gives the desired statement. \square

When in the proof of [14, Corollary 4.4] (respectively, [14, Corollary 2.11]), [14, Proposition 4.2 and Corollary 4.3] (respectively, [14, Proposition 2.7 and Corollary 2.10]) are replaced with Corollary 2.11 and Corollary 2.6, respectively, and having in mind Proposition 3.2, the arguments in those results remain valid to prove:

Corollary 3.4. *Every (not necessarily linear) 2-local triple homomorphism from a Type I_2 JBW*-algebra into a JB*-triple is linear and a triple homomorphism.* \square

The first main result of this note is a consequence of Theorem 3.3, Corollary 3.4 and Corollary 2.6.

Theorem 3.5. *Every (not necessarily linear) 2-local triple homomorphism from a JBW^* -algebra into a JB^* -triple is linear and a triple homomorphism. \square*

The next corollary is interesting by itself.

Corollary 3.6. *Every (not necessarily linear) 2-local triple homomorphism from a von Neumann algebra into a JB^* -triple is linear and a triple homomorphism.*

Since every $*$ -homomorphism between C^* -algebras (respectively, every Jordan $*$ -homomorphism between JB^* -algebras) is a triple homomorphism, Theorems 2.12 and 4.5 and Corollary 4.6 in [14] are direct consequences of the previous Theorem 3.5 and Corollary 3.6.

3.2. 2-local triple homomorphisms on a JBW^* -triple. The rest of the note is devoted to prove the second main result of the paper, in which we shall show that every (not necessarily linear) 2-local triple homomorphism from a JBW^* -triple into a JB^* -triple is linear and a triple homomorphism. The first step toward our goal is the following corollary.

Corollary 3.7. *Let $T : E \rightarrow F$ be a (not necessarily linear) 2-local triple homomorphism from a JBW^* -triple into a JB^* -triple. Then, for each tripotent e in E , $T|_{E_2(e)} : E_2(e) \rightarrow F$ is linear and a triple homomorphism.*

Proof. Clear from Theorem 3.5. \square

We are now in a position to establish the goal of this section.

Theorem 3.8. *Every (not necessarily linear) 2-local triple homomorphism from a JBW^* -triple into a JB^* -triple is linear and a triple homomorphism.*

Proof. Let x, y be two (arbitrary) elements in E . Find a complete tripotent e in E such that $x \in E_2(e)$ (the existence of such a tripotent is guaranteed by [25, Lemma 3.12(1)]). If we write $y = P_2(e)y + P_1(e)y$, then Proposition 2.12 and Corollary 3.7 prove that

$$\begin{aligned} T(x + y) &= T(x + P_2(e)y + P_1(e)y) = T(x + P_2(e)y) + T(P_1(e)y) \\ &= T(x) + T(P_2(e)y) + T(P_1(e)y). \end{aligned}$$

A new application of Proposition 2.12 shows that $T(P_2(e)y) + T(P_1(e)y) = T(P_2(e)y + P_1(e)y)$, and hence $T(x + y) = T(x) + T(y)$. \square

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